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Nature of matrix elements in the quantum chaotic domain of interacting particle systems

V.K.B. Kota[a, 1] and R. Sahu[b]
[a] *Physical Research Laboratory, Ahmedabad 380 009, India*
[b] *Physics Department, Berhampur University, Berhampur 760 007, India*

ABSTRACT: There is a newly emerging understanding that in the chaotic domain of isolated finite interacting many particle systems smoothed densities define the statistical description of these systems and these densities follow from embedded (two-body) random matrix ensembles and their various deformations. These ensembles predict that the smoothed form of matrix elements of a transition operator between the chaotic eigenstates weighted by the densities at the two ends (i.e. the bivariate strength density) will be a bivariate Gaussian with the bivariate correlation coefficient arising out of the non-commutability of the hamiltonian and the transition operator involved. The ensemble theory extends to systems with a mean-field and a chaos generating two-body interaction (as in nuclei, atoms and diffusive quantum dots). These developments in many-body quantum chaos are described with special reference to one-body transition operators.

1. Introduction

Random matrix physics, started with Wigner and Dyson's GOE, GUE and GSE hamiltonian random matrix ensembles [1], has grown far and wide in the last fifteen years with the investigation of a variety of random matrix ensembles (interpolating, banded, partitioned, chiral etc.) and applications in almost all branches of physics [2] (most recent application being in Econophysics [3]). As Guhr et al [2] state 'the universal validity of random matrix theories is reminiscent of the success of thermodynamics in the last century $\cdots$'. Recently it is recognized by large number of research groups in atomic, molecular, nuclear and mesoscopic physics (see for example [4, 5, 6, 7]), in the context of many-body quantum chaos, that embedded random matrix ensembles (EE) (in particular EGOE(k), the embedded Gaussian orthogonal ensemble of random matrices of k-body interactions) are relavant for 'isolated finite interacting particle systems'. Our purpose in this article is to discuss sum of the results of EE for transition matrix elements. In Section 2 definition and construction of EGOE(2) for fermion systems are given; introduced here are also various deformed EGOE. The EGOE(k) results for density of states and transition matrix elements are briefly described in Section 3. The topic 'quantum chaos and transition strength sums' form Section 4. In Section 5, EE(1+2) results for matrix elements of one-body transition operators with emphasis on the role of the mean-field basis and strength functions are presented. Section 6 gives concluding remarks.

2. Embedded Ensembles

EGOE(k) for many fermion systems is defined by assuming that the many particle space is a direct product space, of single particle states, as in the nuclear/atomic shell model. EGOE(k)

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for m ($m > k$) fermion systems (with the particles distributed say in N single particle states) is generated by defining the hamiltonian H, which is say k-body, to be GOE in k-particle space and then propagating it to m - particle spaces by using the geometry of the m - particle spaces [1]. Let us consider EGOE(2). Given the single particle states $|\nu_i\rangle$, $i = 1, 2, \ldots, N$, operator form of the 2-body hamiltonian is (with a in $\langle \, | \, | \, \rangle_a$ indicating that the states involved are antisymmetrized),

$$H = \sum_{\nu_i<\nu_j,\, \nu_k<\nu_l} \langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a \; a^\dagger_{\nu_l} \, a^\dagger_{\nu_k} \, a_{\nu_i} \, a_{\nu_j} \tag{1}$$

Symmetries for the two-body matrix elements (TBME) $\langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a$ in (1) are,

$$\langle \nu_k \, \nu_l \mid H \mid \nu_j \, \nu_i \rangle_a = - \langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a \quad , \qquad \langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a = \langle \nu_i \, \nu_j \mid H \mid \nu_k \, \nu_l \rangle_a \tag{2}$$

The hamiltonian H in m-particle spaces is defined in terms of the TBME via the direct product structure. The non-zero matrix elements of the m-particle H matrix are of three types,

$$\begin{aligned} \langle \nu_1 \nu_2 \cdots \nu_m \mid H \mid \nu_1 \nu_2 \cdots \nu_m \rangle_a &= \sum_{\substack{\nu_i<\nu_j<\nu_m \\ \nu_m}} \langle \nu_i \nu_j \mid H \mid \nu_i \nu_j \rangle_a \\ \langle \nu_p \nu_2 \nu_3 \cdots \nu_m \mid H \mid \nu_1 \nu_2 \cdots \nu_m \rangle_a &= \sum_{\nu_i=\nu_2} \langle \nu_p \nu_i \mid H \mid \nu_1 \nu_i \rangle_a \\ \langle \nu_p \nu_q \nu_3 \cdots \nu_m \mid H \mid \nu_1 \nu_2 \nu_3 \cdots \nu_m \rangle_a &= \langle \nu_p \nu_q \mid H \mid \nu_1 \nu_2 \rangle_a \end{aligned} \tag{3}$$

EGOE(2) is defined by (1)-(3) with GOE representation for H in two-particle space,

$$\begin{aligned} &\langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a \;\; \text{are independent Gaussian random variables} \\ &\overline{\langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a} = 0, \quad \overline{\left| \langle \nu_k \, \nu_l \mid H \mid \nu_i \, \nu_j \rangle_a \right|^2} = v^2 (1 + \delta_{(ij),(kl)}) \end{aligned} \tag{4}$$

In (4) bar denotes ensemble average and v is a constant. Note that the $H(m)$ matrix dimension d is $d(N, m) = \left(\begin{array}{c} N \\ m \end{array} \right)$ and the number of independent matrix elements (ime) are $ime(N) = d_2(d_2+1)/2$ where the two-particle space dimension $d_2 = N(N-1)/2$. For example, $d(12, 6) = 924$ and $ime(12) = 2211$. The EGOE(2) is also called TBRE (two-body random matrix ensemble). Extension of (1-4) for boson systems is straightforward and it is described elsewhere [8]. Hamiltonians for many interacting particle systems contain a mean-field part (one-body part h) and a two-body residual interaction V mixing the configurations built out of the distribution of particles in the mean-field single particle orbits; h is defined by single particle energies (SPE) ϵ_i, $i = 1 - N$ and V is defined by TBME. Then it is more realistic to use EE(1+2), the embedded ensemble of (1+2)-body hamiltonians,

$$\text{EE(1+2)} : \; \{H\} = [h(1)] + \lambda \, \{V(2)\} \quad . \tag{5}$$

Here $\{V\}$ is EE(2), i.e. it is EGOE(2) with $v = 1$ in (4) or an ensemble with TBME being independent random variables with a distribution different from Gauassian (for example uniform distribution). Similarly $[h]$ is a fixed hamiltonian or an ensemble with SPE chosen random but following some distribution. Finally, $[h]$ and $\{V\}$ are independent. It is to be expected that the generic features of EE(1+2) approach those of EGOE(k) for sufficiently large values of λ and significant results emerge as λ is varied starting from $\lambda = 0$. Let us mention that EE(1+2) is also called TBRIM (two-body random interaction model). A second class of EE are the partitioned embedded ensembles (p-EE) where the hamiltonian in two-particle space is block structured (i.e.

the space divides into subspaces $\sum \oplus\Gamma$) with variances of the matrix elements in each block being $v^2_{\Gamma\Gamma'}$ for the block connecting Γ and Γ' subspaces. Third class of EE are EE-sym where $v^2_{\Gamma\Gamma'} = 0$ for $\Gamma \neq \Gamma'$ and these ensembles are important for Hamiltonians carrying symmetries (for example J or JT in nuclei and J or LST in atoms). Finally there are also the modified K+EE ensembles where K is a fixed operator.

3. Basic results for EGOE(k)

Generic results for state densities and transition matrix elements, for EGOE(k), that are essentially valid in the dilute limit, which correspond to $N, m, k \to \infty$, $m/N \to 0$ and $k/m \to 0$, are well known. The eigenvalue density $I(E)$ or its normalized version $\rho(E)$ takes Gaussian form [9],

$$I(E) = \langle\langle \delta(H-E)\rangle\rangle = d\,\rho(E)\;; \quad \rho(E) \stackrel{EGOE}{\longrightarrow} \overline{\rho(E)} = \rho_{\mathcal{G}}(E) = \frac{1}{\sqrt{2\pi}\sigma} exp - \frac{1}{2}\left(\frac{E-\epsilon}{\sigma}\right)^2 \tag{6}$$

The binary correlation approximation, originally used by Wigner for deriving the semi-circle state density for GOE ($k = m$ in EGOE(k)) is used by Mon and French [9] to derive (6) via the m-particle space moments $\langle H^p\rangle^m$ of $I(E)$. Firstly it is seen that by definition all odd moments of $I(E)$ will vanish. Using the normalization that $\langle H^2\rangle^{m=2} = 1$ (then $\langle H^2\rangle^m = \binom{m}{k}$) one has the basic result $\overline{H(k)O(t)H(k)} \Rightarrow \binom{m-t}{k} O(t)$ under ensemble average in m-particle spaces and in the dilute limit. Using this and that in the trace $\langle H^p\rangle^m$ binary associations dominate, one can drive formulas for the moments by writing down all possible binary association diagrams. For example for $\langle H^4\rangle$ there are three diagrams, $AABB \oplus ABBA \oplus ABAB \Rightarrow 2\{AABB\} \oplus ABAB$. Note that A and B are H-operators and they are k-body in nature. Evaluating the irreducible diagrams $AABB$ and $ABBA$ give the 4th reduced moment μ_4 and the 4th cumulant k_4,

$$k_4 = \mu_4 - 3 = \left\{\left\langle H^2\right\rangle^m\right\}^{-2} \left\langle H^4\right\rangle^m \; - 3 = \binom{m-k}{k}\binom{m}{k}^{-1} - 1 \stackrel{k<<m}{\longrightarrow} -k^2/m \tag{7}$$

Similar results for the 6th and 8th cumulants are derived: $k_6 = k^3(6k-1)/m^2 + O(1/m^3)$ and $k_8 = -4k^5(23k-9)/m^3 + O(1/m^4)$. Thus in the dilute limit one recovers the Gaussian form for state densities (note that we need in fact not $k/m \to 0$ but $k^2/m \to 0$). Thus, for a two-body interaction $m \sim 12$ gives a good Gaussian. Note that for $m = 4$ one has $k_4 = -1$ implying semi-circle shape as seen in many numerical calculations. In practice one has to apply Edgeworth (mostly 3rd and 4th moment/cumulant) corrections to the Gaussian form. More important point, though not discussed here, is that local level fluctuations given by EGOE(k) are of GOE type.

Similar to (6), for EGOE(k) the transition matrix elements weighted by the densities at the two ends (i.e. bivariate strength densities) take bivariate Gaussian form [10],

$$I_{biv;\mathcal{O}}(E,E') = \left\langle\left\langle \mathcal{O}^\dagger \delta(H-E')\mathcal{O}\delta(H-E)\right\rangle\right\rangle$$

$$= I'(E')\,|\langle E' \mid \mathcal{O} \mid E\rangle|^2\, I(E) = \left\langle\left\langle \mathcal{O}^\dagger\mathcal{O}\right\rangle\right\rangle\; \rho_{biv;\mathcal{O}}(E,E')\;;$$

$$\rho_{biv;\mathcal{O}}(E,E') \stackrel{EGOE}{\longrightarrow} \overline{\rho_{biv;\mathcal{O}}(E,E')} = \rho_{biv-\mathcal{G};\mathcal{O}}(E,E'\;;\;\epsilon_1,\epsilon_2,\sigma_1,\sigma_2,\zeta) =$$

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\zeta^2}}\, exp\left\{-\frac{1}{2(1-\zeta^2)}\left[\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 - 2\zeta\left(\frac{E-\epsilon_1}{\sigma_1}\right)\left(\frac{E'-\epsilon_2}{\sigma_2}\right) + \left(\frac{E'-\epsilon_2}{\sigma_2}\right)^2\right]\right\} \tag{8}$$

The bivariate reduced central moments of $\rho_{biv;\mathcal{O}}$ are $\mu_{pq} = \left\langle \mathcal{O}^\dagger \left(\frac{H-\epsilon_2}{\sigma_2}\right)^q \mathcal{O} \left(\frac{H-\epsilon_1}{\sigma_1}\right)^p \right\rangle \Big/ \left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle$ and $\zeta = \mu_{11}$ is the bivariate correlation coefficient. Let us consider the evaluation of μ_{pq} by representing H by EGOE(k) and the transition operator $\mathcal{O}$ by EGOE(t). It is also assumed that H and $\mathcal{O}$ ensembles are independent. Now the correlations in μ_{pq} arise due to the non-commutability of H and $\mathcal{O}$ operators. Firsly it is seen that all μ_{pq} with $p+q$ odd will vanish on ensemble average and also $\mu_{pq} = \mu_{qp}$. Moreover $\sigma_1^2 = \sigma_2^2 = \left\langle \mathcal{O}^\dagger \mathcal{O} H^2 \right\rangle / \left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle = \left(\begin{array}{c} m \\ k \end{array} \right)$. The first non-trivial moment μ_{11} is,

$$\begin{aligned} \zeta = \mu_{11} &= \left\langle O^\dagger(t) H(k) O(t) H(k) \right\rangle^m / \left\{ \left\langle O^\dagger(t) O(t) \right\rangle^m \langle H(k) H(k) \rangle^m \right\} \rightarrow ABAB/(AA)(BB) \\ &= \left(\begin{array}{c} m-t \\ k \end{array} \right) \left(\begin{array}{c} m \\ k \end{array} \right)^{-1} = 1 - \frac{kt}{m} + \frac{k(k-1)t(t-1)}{2m^2} \quad + O(1/m^3) \end{aligned} \tag{9}$$

Let us now consider the cases with $p+q=4$. The diagrams for these follow by putting $\mathcal{O}^\dagger$ and $\mathcal{O}$ at appropriate places in the $\langle H^4 \rangle$ diagrams,

$$\begin{aligned} \mu_{04} &= \left[\left\langle \mathcal{O}^\dagger AABB\mathcal{O} \right\rangle \oplus \left\langle \mathcal{O}^\dagger ABBA\mathcal{O} \right\rangle \oplus \left\langle \mathcal{O}^\dagger ABAB\mathcal{O} \right\rangle \right] / \left[\left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle \left(\langle H^2 \rangle \right)^2 \right] \\ &= 2 + \left(\begin{array}{c} m-k \\ k \end{array} \right) \left(\begin{array}{c} m \\ k \end{array} \right)^{-1} \\ \mu_{13} &= \left[\left\langle \mathcal{O}^\dagger AAB\mathcal{O}B \right\rangle \oplus \left\langle \mathcal{O}^\dagger ABB\mathcal{O}A \right\rangle \oplus \left\langle \mathcal{O}^\dagger ABA\mathcal{O}B \right\rangle \right] / \left[\left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle \left(\langle H^2 \rangle \right)^2 \right] \\ &= \left[2 \left(\begin{array}{c} m-t \\ k \end{array} \right) \left(\begin{array}{c} m \\ k \end{array} \right) + \left(\begin{array}{c} m-k \\ k \end{array} \right) \left(\begin{array}{c} m-t \\ k \end{array} \right) \right] \left(\begin{array}{c} m \\ k \end{array} \right)^{-2} \\ \mu_{22} &= \left[\left\langle \mathcal{O}^\dagger AA\mathcal{O}BB \right\rangle \oplus \left\langle \mathcal{O}^\dagger AB\mathcal{O}BA \right\rangle \oplus \left\langle \mathcal{O}^\dagger AB\mathcal{O}AB \right\rangle \right] / \left[\left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle \left(\langle H^2 \rangle \right)^2 \right] \\ &= \left[\left(\begin{array}{c} m \\ k \end{array} \right)^2 + \left(\begin{array}{c} m-t \\ k \end{array} \right)^2 + \left(\begin{array}{c} m-k-t \\ k \end{array} \right) \left(\begin{array}{c} m-t \\ k \end{array} \right) \right] \left(\begin{array}{c} m \\ k \end{array} \right)^{-2} \\ k_{04} &= k_{40} = \mu_{40} - 3 = \left(\begin{array}{c} m-k \\ k \end{array} \right) \left(\begin{array}{c} m \\ k \end{array} \right)^{-1} - 1 = -\frac{k^2}{m} + \frac{k^2(k-1)^2}{2m^2} \quad + O(1/m^3) \\ k_{13} &= k_{31} = \mu_{31} - 3\mu_{11} = \zeta k_{04} \\ k_{22} &= \mu_{22} - 2\zeta^2 - 1 = \zeta^2 \left\{ \left(\begin{array}{c} m-k-t \\ k \end{array} \right) \left(\begin{array}{c} m-t \\ k \end{array} \right)^{-1} - 1 \right\} \\ &= -\frac{k^2}{m} + \frac{k^2 \left[(k-1)^2 + 4kt - 2t \right]}{2m^2} \quad + O(1/m^3) \end{aligned} \tag{10}$$

Further results and numerical tests etc. are given in [10]. Finally it is worth mentioning that the EGOE smoothed Gaussian forms for the state densities (6) and the bivariate strength densities (8) together with the result that local strength fluctuations for EGOE (just as for GOE) are of Porter-Thomas type, are used recently to derive the EGOE formulas for information entropy (S^{info}) and number of principal components (NPC), which are measures of complexity and chaos, in transition strength distributions [5].

4. EE(1+2) and quantum chaos: transition strength sums

The principal reason for EGOE receiving considerable attention in the last 3-4 years is due to the fact that: *there is a newly emerging understanding that in the chaotic domain of isolated finite*

interacting many particle systems smoothed densities (they include strength functions) define the statistical description of these systems and these densities follow from embedded random matrix ensembles. This conjucture is based on the calculations **for transition strength sums**

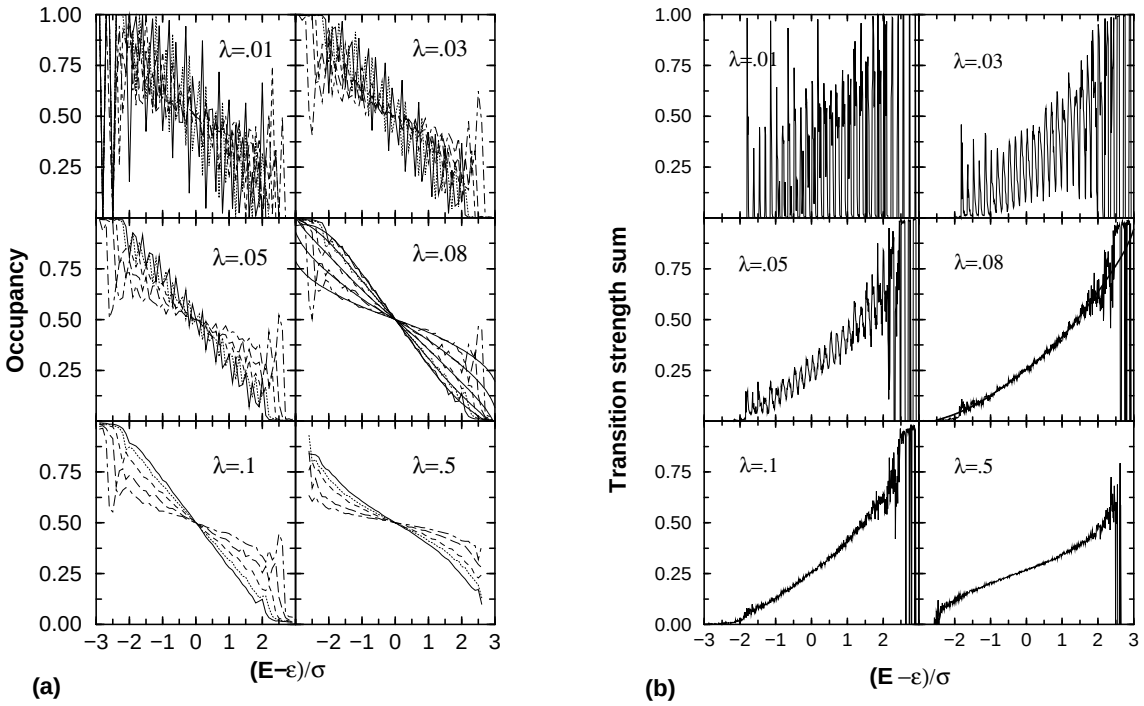

Fig. 1. (a) Occupation numbers for a 25 member EE(1+2) ensemble, defined by the hamiltonian $h(1) + \lambda\{V(2)\}$, in the 924 dimensional $N = 12$, $m = 6$ space. Results are shown for the lowest 5 single particle states and for six values of λ. In the calculations occupation numbers are averaged over a bin size of 0.1 in $(E - \epsilon)/\sigma$; ϵ is centroid and σ is width. The spectra of all the ensemble members are first zero centered and scaled to unit width and then the ensemble average is carried out. The estimate of [11] gives $\lambda_c \sim 0.05$ for order-chaos border in the present EE(1+2) example. It is clearly seen that once chaos sets in, the occupation numbers take stable smoothed forms. For $\lambda = .08$, the smooth forms are well represented by ratio of Gaussians (smooth curves in the figure). **(b)** same as (a) but for the transition strength sum generated by the one-body transition operator $a_2^\dagger a_9$. The bin size used here is 0.01.

carried out using different types of interpolating hamiltonians generating order-chaos transitions in many different systems: (i) for occupancies using a 20 member EE(1+2) in 330 dimensional $N = 11, m = 4$ space with $h(1)$ defined by the single particle enegies $\epsilon_i = i + (1/i)$; $i = 1, 2, \ldots, 11$ and $V(2)$ is EGOE(2) [12]; (ii) for occupancies using the four interacting electrons Ce atom [4]; (iii) for occupancies using a symmetrized coupled two-rotor model [13]; (iv) using two different types of interpolating ensembles in nuclear shell model for ^{24}Mg for the Gamow-Teller strength sums and for occupancies [6]. Most significant conclusion of all these studies is that transition strength sums (note that occupancies are one-particle transfer strength sums) show quite different behaviour in regular and chaotic domains of the spectrum. In order to make this argument clear, calculations are carried out using EE(1+2) as in (i) but for $N = 12$ and $m = 6$. The EGOE gives

via (8) that the transition strength sum density $I_{\mathcal{O}^\dagger\mathcal{O}}(E) = \Big\langle\Big\langle \mathcal{O}^\dagger\mathcal{O}\delta(H-E)\Big\rangle\Big\rangle$, which is a marginal density of the bivariate strength density, is a Gaussian. Therefore, using (6), it is immediately seen that transition strength sums vary with excitation energy as ratio of Gaussians. Fig. 1a shows results for occupancies and Fig. 1b for transition strength sums for the one-body operator $a_2^\dagger a_9$ calculated for various values of the interpolating parameter λ in the EE(1+2) hamiltonian. From Figs. 1a,b it is clearly seen that below the region of onset of chaos transition strength sums show strong fluctuations (in the regular ground state domain perturbation theory applies). In this region there is no equilibrium distribution for $I_{\mathcal{O}^\dagger\mathcal{O}}(E)$, $I(E)$ and other densities. However in the chaotic domain (there are methods for determining the critical λ_c that marks order-chaos border [6, 11]) the densities can be replaced by their smoothed forms. Note that fluctuations in strength sums are basically given by 1/NPC. Therefore there is a statistical mechanics, defined by various smoothed densities and they are given by EGOE, operating in the quantum chaotic domain of isolated finite interacting particle systems (this is also the essence of statistical nuclear spectroscopy [10, 14, 15] and also statistical spectroscopy in atoms [4]); see also [6]. In fact in favourable situations, it is possible to introduce thermodynamic concepts (effective temperatures and chemical potentials etc.) in the chaotic domain [4].

5. EE(1+2) results for matrix elements of one-body transition operators: role of the mean-field basis and strength functions

5.1 Strength functions: transition from Breit - Wigner to Gaussian form

Decomposing the m-particle spaces into subspaces Γ (say defined by the irreps of a group structure in the Hilbert space defined by (N,m)), i.e. $m \rightarrow \sum\ \Gamma$, gives,

$$\begin{aligned} I(E) = \langle\langle \delta(H-E)\rangle\rangle = \sum_\Gamma \langle\langle \delta(H-E)\rangle\rangle^\Gamma = \sum_\Gamma I^\Gamma(E)\ ;\ \ I^\Gamma(E) \longrightarrow \overline{I^\Gamma(E)} = I^\Gamma_{\mathcal{G}}(E) \\ I_K(E) = \langle\langle K\delta(H-E)\rangle\rangle = \sum_\Gamma \langle\langle K\delta(H-E)\rangle\rangle^\Gamma = \sum_\Gamma I^\Gamma_K(E)\ ;\ \ I^\Gamma_K(E) \longrightarrow \overline{I^\Gamma_K(E)} = I^\Gamma_{K:\mathcal{G}}(E) \end{aligned} \tag{11}$$

There are plausible arguments in favour of the Gaussian forms in (11) for EGOE. The decompositions in (11) are related to the well known 'strength functions' (also called local spectral density of states (LDOS) in literature) which are basic ingredients of a many particle system. Given a compound state ϕ_k, the probability of its decay into stationary states ψ_E (generated by H) is given by $|\langle\phi_k \mid \psi_E\rangle|^2$. Then the strength function $F_k(E)$ is,

$$|\,\phi_k\rangle = \sum_E\ C^E_k\ |\,\psi_E\rangle \quad ; \quad F_k(E) = \sum_{E'}\ \Big|C^{E'}_k\Big|^2\ \delta(E-E') = \langle\delta(H-E)\rangle^k \tag{12}$$

If we consider a set of states ϕ_k that belong to a irrep Γ as in (11), then the average strength function $F_\Gamma(E)$ is nothing but the partial densities $I_\Gamma(E)$ in (11); $I_\Gamma(E) = \sum_{k\varepsilon\Gamma}\ F_k(E)$ and $F_\Gamma(E) = I_\Gamma(E)/d(\Gamma)$. A quite different and useful way to look at strength functions is to think of ϕ_k as a compound state generated by the action of a transition operator $\mathcal{O}$ on a state ψ_{E_k} (for example ground state),

$$\begin{aligned} |\,\phi_{E_k}\rangle = \frac{\mathcal{O}\ |\,\psi_{E_k}\rangle}{\left[\langle\psi_{E_k} \mid \mathcal{O}^\dagger\mathcal{O} \mid \psi_{E_k}\rangle\right]^{1/2}}\ \ ;\ \ \langle\phi_{E_k} \mid \phi_{E_k}\rangle = 1 \\ |\,\phi_{E_k}\rangle = \sum_E\ C^E_{E_k}\ |\,\psi_E\rangle \quad \Rightarrow \quad F_{k;\mathcal{O}}(E) = \frac{\Big\langle\Big\langle \mathcal{O}^\dagger\delta(H-E)\mathcal{O}\delta(H-E_k)\Big\rangle\Big\rangle}{\langle\langle \mathcal{O}^\dagger\mathcal{O}\delta(H-E_k)\rangle\rangle} \end{aligned} \tag{13}$$

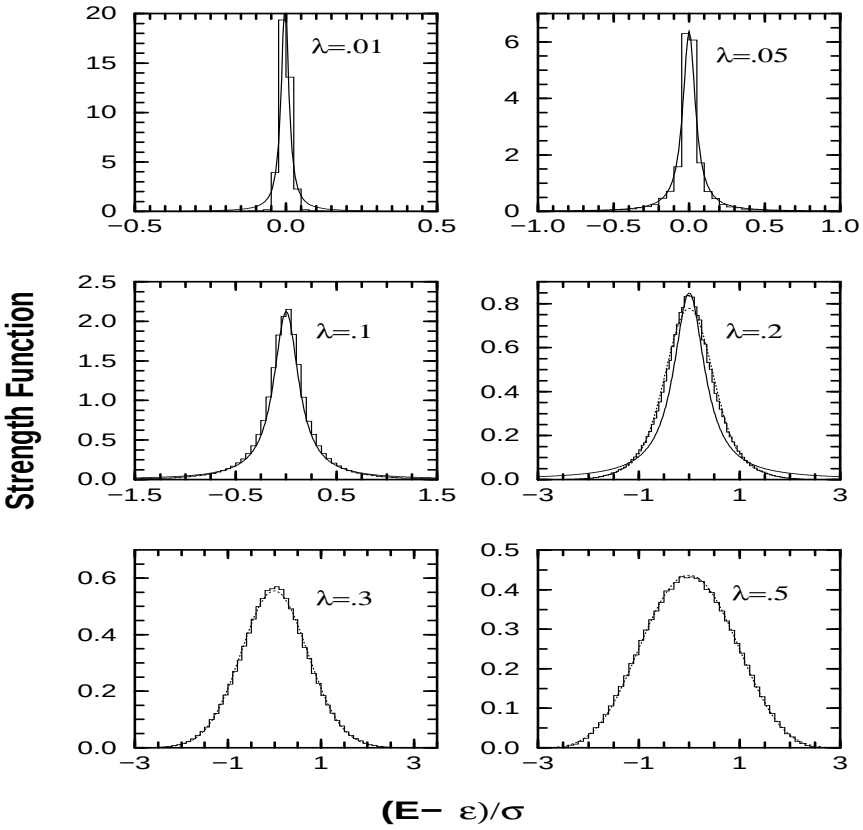

Fig. 2. Strength functions ($F_{\mathcal{E}_k}(E)$) for the EGOE(1+2) ensemble defined in Fig. 1. The $|k\rangle$ states are the mean-field $h(1)$ states defined by the distribution of m particles in the N single particle states and their energies $\mathcal{E}_k$ are $\mathcal{E}_k = \langle k|h(1) + \lambda\, V(2)|k\rangle$. In the calculations E and $\mathcal{E}_k$ are zero centered for each member and scaled by the spectrum (E's) width σ; $\hat{E}_k = (\mathcal{E}_k - \epsilon)/\sigma$ and $\hat{E} = (E - \epsilon)/\sigma$. For each member $|C_k^E|^2$ are summed over the basis states $|k\rangle$ in the energy window $\hat{E}_k \pm \Delta$ and then ensemble averaged $F_{\hat{E}_k}(\hat{E})$ vs $\hat{E}$ is constructed as a histogram; the value of Δ is chosen to be 0.025 for $\lambda < 0.1$ and beyond this $\Delta = 0.1$. Results are shown for $\lambda = 0.01$, 0.05, 0.1, 0.2, 0.3 and 0.5. The histograms are EGOE(1+2) results, continuous curves are BW fit and dotted curves are Edgeworth corrected Gaussians (ED); the ED incorporates [16] skewness (γ_1) and excess (γ_2) corrections. For $\lambda = 0.01$, 0.05 and 0.1 only BW, for 0.2 both BW and ED and for 0.3 and 0.5 only ED are shown. In the figure all the results are for $\hat{E}_k = 0$.

Starting with the EE(1+2), it is to be expected that for sufficiently large values of λ in (5), EGOE(k) description should be valid and therefore applying (8) gives the shape of the strength function to be Gaussian (conditional density of a bivariate Gaussian is a Gaussian) and hence its width σ_k is independent of the energy E_k or k. These results are indeed seen in numerical calculations. The important question is how $F_k(E)$ changes as λ is varied. The standard form, normally employed in many applications, for strength functions is the Breit-Wigner (BW) form characterized by a spreading width Γ_k,

$$F_{k:BW}(E) = \frac{1}{2\pi}\, \frac{\Gamma_k}{(E - \overline{E_k})^2 + \Gamma_k^2/4} \tag{14}$$

where $\overline{E_k} = \langle k \mid H \mid k\rangle = \int_{-\infty}^{\infty} F_k(E) E\, dE$. It is expected that BW form and the Gaussian form should appear as λ is varied and this question is investigated in: (i) using EE(1+2) and constructing $F_k(E)$ for various values of λ, here ϕ_k are the mean-field (h) basis states (only final result of this

study is reported in [12]); (ii) by carrying out nuclear shell model calculations with $H_\lambda = h + \lambda V$ for ^{28}Si with ϕ_k chosen to be mean-field basis states [17]; (iii) using the three-orbital Lipkin-Meshkov-Glick model [18]; (iv) using a symmetrized coupled two-rotor model [13]. Results from these studies show that when the system is chaotic (so that level and strength fluctuations follow from GOE), strength functions take Gaussian form and order-chaos transition implies for $F_k(E)$ transition from BW to Gaussian,

$$order \rightarrow chaos \quad \Rightarrow \quad F_{k:BW}(E) \longrightarrow F_{k:\mathcal{G}}(E) \tag{15}$$

For further understanding of the nature of $F_k(E)$ we performed EGOE(1+2) calculations in the 924 dimensional $N = 12$ and $m = 6$ space and the results are shown in Fig. 2. From the figure it is clearly seen that there is BW to Gaussian transition and the value of the interpolating parameter $\lambda = \lambda_{F_k}$ for onset of this transition is $\lambda_{F_k} \approx 0.2$. This should be compared with $\lambda = \lambda_c \approx 0.08$ for onset of chaos in level fluctuations, occupancies and strength sums of one-body operators in the present EGOE(1+2) example (Fig. 1). A similar BW to Gaussian transition is seen in [17, 18] and in the nuclear shell model example considered in [17] $\lambda_c \sim 0.3$ and $\lambda_{F_k} \sim 0.6$. Thus the BW form for $F_k(E)$ extends into the chaotic domain ($\lambda > \lambda_c$) and the transition to Gaussian shape takes place in the second chaotic layer defined by $\lambda > \lambda_{F_k}$. Full understanding of λ_{F_k} is at present lacking.

5.2 matrix elements of one-body transition operators in the chaotic domain

Let us consider a one-body transition operator written in occupation number representation $\mathcal{O} = \epsilon_{\alpha\beta} a^\dagger_\alpha a_\beta$ and its matrix elements $|\langle E_f \mid \mathcal{O} \mid E_i \rangle|^2$ in many-particle eigenstates expanded in the mean-field basis states $|k_i\rangle$ are (with ϵ_α being single particle energies and $\mathcal{E}_i$ being the mean-field basis states energies),

$$|\langle E_f \mid \mathcal{O} \mid E_i \rangle|^2 = \left\{ \sum_{k_i\, k_f} C^{E_i}_{k_i}\, C^{E_f}_{k_f}\ \langle k_f \mid \mathcal{O} \mid k_i \rangle \right\}^2 = diag + offdiag$$
$$= \sum_{k_i\, k_f} \left| C^{E_i}_{k_i} \right|^2 \left| C^{E_f}_{k_f} \right|^2 |\langle k_f \mid \mathcal{O} \mid k_i \rangle|^2 + \sum_{k_i \neq k'_i\, ,\, k_f \neq k'_f} C^{E_i}_{k_i}\, C^{E_i}_{k'_i}\, C^{E_f}_{k_f}\, C^{E_f}_{k'_f}\ \langle k_f \mid \mathcal{O} \mid k_i \rangle \left\langle k'_f \mid \mathcal{O} \mid k'_i \right\rangle \tag{16}$$

The *diag* term in (16) involves only the strength functions (see (12)) and also for this term, for a given k_f and k_i only one $\epsilon_{\alpha\beta}$ in $\mathcal{O}$ will contribute. Following [15] it is easily seen that,

$$|\langle E_f \mid \mathcal{O} \mid E_i \rangle|^2_{diag} = \sum_{\alpha\, \beta} |\epsilon_{\alpha\beta}|^2 \ \left\{ \sum_{\mathcal{E}_i} \langle n_\beta (1 - n_\alpha) \rangle^{\mathcal{E}_i} \left| C^{E_i}_{\mathcal{E}_i} \right|^2 \ \left| C^{E_f}_{\mathcal{E}_f = \mathcal{E}_i - \epsilon_\beta + \epsilon_\alpha} \right|^2 \right\} \tag{17}$$

Assuming that $\langle n_\beta (1 - n_\alpha) \rangle^{\mathcal{E}_i}$ do not vary much over the number of principal components (say N_{eff} is NPC for the mean-field basis states), it can be replaced by its corresponding average (this is verified using EE(1+2) in [19]). Then,

$$|\langle E_f \mid \mathcal{O} \mid E_i \rangle|^2_{diag} = \sum_{\alpha\, \beta} |\epsilon_{\alpha\beta}|^2 \ \langle n_\beta (1 - n_\alpha) \rangle^{E_i} \ \left\{ \sum_{\mathcal{E}_i} \left| C^{E_i}_{\mathcal{E}_i} \right|^2 \ \left| C^{E_f}_{\mathcal{E}_f = \mathcal{E}_i - \epsilon_\beta + \epsilon_\alpha} \right|^2 \right\} \tag{18}$$

The $|C|^2$'s in (18) are nothing but strength functions. With the assumption that strength functions are a function of $(E - \mathcal{E}_i)/s_i$ where s_i is a scale parameter (spreading width Γ_i for BW and the

spectral width σ_i for Gaussian), we have the results (with $\overline{D(E)}$ denoting mean spacing),

$$\begin{aligned} F_k(E) &= (s_k)^{-1} f((E-\mathcal{E}_k)/s_k)\ , \quad \int F_k(E)dE = 1 \\ \left|C_k^E\right|^2 &= \overline{D(E)}\ F_k(E)\ \Rightarrow (N_{eff}f(0))^{-1}\ f((E-\mathcal{E}_k)/s_k)\ , \quad \overline{D(E)} = [N_{eff}f(0)]^{-1}\ \overline{s} \\ \sum_k \left|C_k^E\right|^2 &\longrightarrow \int \left|C_k^E\right|^2 \frac{d\mathcal{E}_k}{\overline{D(E)}} = \int F_k(E)d\mathcal{E}_k \end{aligned} \tag{19}$$

Substituting F's for $|C|^2$ in (18) and replacing the $\mathcal{E}_i$ summation by an integral, both using (19), give the final result in terms of occupancies and the mean spacings,

$$\begin{aligned} |\langle E_f \mid \mathcal{O} \mid E_i\rangle|^2_{diag} &= \sum_{\alpha,\beta} |\epsilon_{\alpha\beta}|^2\ \langle n_\beta(1-n_\alpha)\rangle^{E_i}\ \overline{D(E_f)}\ \int F_{\mathcal{E}_i}(E_i)\ F_{\mathcal{E}_f=\mathcal{E}_i-\epsilon_\beta+\epsilon_\alpha}(E_f)\ d\mathcal{E}_i \\ &= \sum_{\alpha,\beta} |\epsilon_{\alpha\beta}|^2\ \langle n_\beta(1-n_\alpha)\rangle^{E_i}\ \overline{D(E_f)}\ \mathcal{F}(\Delta, s_i, s_f)\ ; \qquad \Delta = E_f - E_i + \epsilon_\beta - \epsilon_\alpha \\ \mathcal{F}(\Delta,\Gamma_i,\Gamma_f)_{BW} &= \frac{1}{2\pi}\frac{\Gamma_i+\Gamma_f}{\Delta^2+(\Gamma_i+\Gamma_f)^2/4} \\ \mathcal{F}(\Delta,\sigma_i,\sigma_f)_{Gauss} &= \frac{1}{\sqrt{2\pi(\sigma_i^2+\sigma_f^2)}}\ exp - \frac{\Delta^2}{2(\sigma_i^2+\sigma_f^2)} \end{aligned} \tag{20}$$

Eq. (20) with $\mathcal{F}_{BW}$ was derived earlier [4, 19]. The procedure used in deriving *diag* term is equivalent to starting with $H = h(1) + V(2)$, first generating the non-interacting paricle (NIP) bivariate strength density $I^h_{biv;\mathcal{O}}(x,x')$ due to $h(1)$ (using the results of [15]), then convoluting the NIP spikes with independent spreading functions, due to $V(2)$, at the two energies and finally simplifying as in (20). This assumes that the spreadings are uncorrelated. However, the $offdiag$ term in (16) arises due to correlations and the bivariate correlation coefficient $\zeta = \left\langle \mathcal{O}^\dagger V \mathcal{O} V \right\rangle / \left\langle \mathcal{O}^\dagger \mathcal{O} \right\rangle \langle VV \rangle$, takes them into account, i.e. the spreading function $\rho^V_{biv;\mathcal{O}}(y,y')$ in

$$I^H_{biv;\mathcal{O}}(E,E') = I^h_{biv;\mathcal{O}} \otimes \rho^V_{biv;\mathcal{O}}[E,E']$$

is not $\rho^V_{\mathcal{G};i}(y)\rho^V_{\mathcal{G};f}(y')$ but it is $\rho^V_{biv;\mathcal{O}}(y,y';0,0,\sigma_i,\sigma_f,\zeta)$. Following [15] it is easily seen that even with ζ the final result is same as in (20) but with a modified $\mathcal{F}$,

$$\mathcal{F}(\Delta,\sigma_i,\sigma_f,\zeta)_{biv-\mathcal{G}} = \frac{1}{\sqrt{2\pi(\sigma_i^2+\sigma_f^2-2\zeta\sigma_i\sigma_f)}}\ exp - \frac{\Delta^2}{2(\sigma_i^2+\sigma_f^2-2\zeta\sigma_i\sigma_f)} \tag{21}$$

Therefore for $\Delta = 0$ and $\zeta \to 1$, there is enhancement in the matrix elements compared to the *diag* approximation. This is indeed seen in EE(1+2) calculations earlier [19] and now it is clear that with ζ in $\rho^V_{biv;\mathcal{O}}$ we have the proper theory for transition matrix elements of one-body operators (the theory with (21) is expected to operate in the chaotic domain defined by $\lambda > \lambda_{F_k}$ and λ_{F_k} is defined in Section 5.1) . In fact for $\sigma_i \sim \sigma_f$, the enhancement is $1/\sqrt{(1-\zeta)}$. Note that for $\zeta = 0$ we get back the *diag* approximation which is a GOE result. Also as noted in [19], the enhancement grows with m as ζ grows with m (see (9)). Applications of the results in (20,21) and their variants [15], for nuclei and atoms are given in [15, 20] and [4, 21] respectively.

6. Conclusions

In this article an attempt is made to give an overview of the subject of embedded random matrix ensembles (classical EGOE and its various deformations) for complexity and chaos in interacting particle systems. It should be clear that generic embedded ensembles results are relevant in the quantum chaotic domain of isolated finite interacting many particle systems such as nuclei, atoms, molecules, atomic clusters, quantum dots etc. and therefore large scale explorations of deformed embedded ensembles are called for.

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